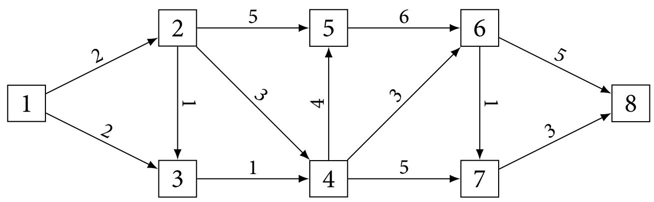
# Lesson 10. The Principle of Optimality and Formulating Recursions

## 0 Warm up

**Example 1.** Consider the following directed graph. The labels on the edges are edge lengths.




In this order:

a. Find a shortest path from node 1 to node 8. What is its length?

Path: $1 \to 3 \to 4 \to 6 \to 7 \to 8$ Length: 10

b. Find a shortest path from node 3 to node 8. What is its length?

Path: $3 \to 4 \to 6 \to 7 \to 8$ Length: 8

c. Find a shortest path from node 4 to node 8. What is its length?

Path: $4 \to 6 \to 7 \to 8$ Length: 7

## 1 The principle of optimality

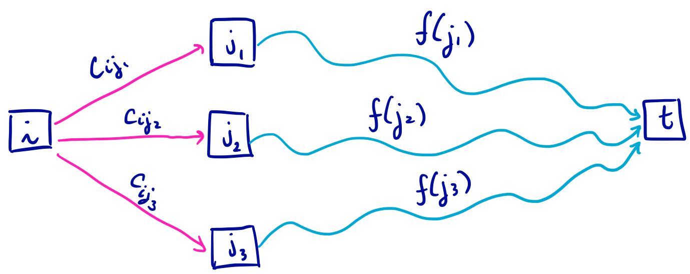
- Let $P$ be the path $1 \to 3 \to 4 \to 6 \to 7 \to 8$ in the graph for Example 1
  - $P$ is a shortest path from node 1 to node 8, and has length 10
- Let $P'$ be the path $3 \to 4 \to 6 \to 7 \to 8$
  - $P'$ is a **subpath** of $P$ with length 8
- Is $P'$ a shortest path from node 3 to node 8?
  - Suppose we had a path $Q$ from node 3 to node 8 with length $< 8$
  - Let $R$ be the path consisting of edge $(1, 3) + Q$
  - Then, $R$ is a path from node 1 to node 8 with length $< 2 + 8 = 10$
  - This contradicts the fact that $P$ is a shortest path from node 1 to node 8
  - Therefore, there cannot be a path from node 3 to node 8 with length $< 8$.

**The principle of optimality (for shortest path problems)**

In a directed graph with no negative cycles, optimal paths must have optimal subpaths.

- How can we exploit this?
- If optimal paths must have optimal subpaths, then we can construct a shortest path by extending known shortest subpaths
- Consider a directed graph $(N, E)$ with target node $t \in N$ and edge lengths $c_{ij}$ for $(i, j) \in E$
- By the principle of optimality, the shortest path from node $i$ to node $t$ must be:

$$\text{edge } (i, j) + \text{shortest path from } j \text{ to } t \qquad \text{for some } j \in N \text{ such that } (i, j) \in E$$




shortest path length: $f(i) = \min\{c_{ij_1} + f(j_1),\ c_{ij_2} + f(j_2),\ c_{ij_3} + f(j_3)\}$

## 2 Formulating recursions

- A **recursion** defines the value of a function in terms of other values of the function
- Let

$$f(i) = \text{length of a shortest path from node } i \text{ to node } t \quad \text{for every node } i \in N$$

- Using the principle of optimality, we can define $f$ recursively by specifying
  (i) the **boundary conditions** and
  (ii) the **recursion**
- The boundary conditions provide a "base case" for the values of $f$:

$f(t)$ = length of a shortest path from node $t$ to node $t$ = 0

- The recursion specifies how the values of $f$ are connected:

$$f(i) = \min_{j \text{ s.t. } (i,j) \in E} \{c_{ij} + f(j)\} \qquad \text{for } i \in N,\ i \neq t$$

**Example 2.** Use the recursion defined above to find the length of a shortest path from nodes 1, …, 8 to node 8 in the graph for Example 1. Use your computations to find a shortest path from node 1 to node 8.

$f(8) =$ $0$

$f(7) =$ $\min\{c_{78} + f(8)\} = \min\{\underset{(7,8)}{3+0}\} = 3$

$f(6) =$ $\min\{c_{67} + f(7),\ c_{68} + f(8)\} = \min\{\underset{(6,7)}{1+3},\ 5+0\} = 4$

$f(5) =$ $\min\{c_{56} + f(6)\} = \min\{6+4\} = 10$

$f(4) =$ $\min\{c_{45} + f(5),\ c_{46} + f(6),\ c_{47} + f(7)\} = \min\{4+10,\ \underset{(4,6)}{3+4},\ 5+3\} = 7$

$f(3) =$ $\min\{c_{34} + f(4)\} = \min\{\underset{(3,4)}{1+7}\} = 8$

$f(2) =$ $\min\{c_{23} + f(3),\ c_{24} + f(4),\ c_{25} + f(5)\} = \min\{1+8,\ 3+7,\ 5+10\} = 9$

$f(1) =$ $\min\{c_{12} + f(2),\ c_{13} + f(3)\} = \min\{2+9,\ \underset{(1,3)}{2+8}\} = 10$

Shortest path from node 1 to node 8:

$(1,3),\ (3,4),\ (4,6),\ (6,7),\ (7,8)$ or $1 \to 3 \to 4 \to 6 \to 7 \to 8$

- Food for thought:
  - Does the order in which you solve the recursion matter?
  - Why did the ordering above work out for us?

## 3 Next lesson...

- Dynamic programs are not usually given as shortest/longest path problems as we have done over the past few lessons
- Instead, dynamic programs are usually given as recursions
- We'll get some practice using this "standard language" to describe dynamic programs